

Homework 3: Duality

Instructions: Submit a single Jupyter notebook (.ipynb) of your work to Canvas by 11:59pm on the due date. All math must be formatted using LaTeX. **Be sure to show all the work involved in deriving your answers! If you just give a final answer without explanation, you may not receive credit for that question.**

You may discuss the concepts with your classmates, look on the internet for help, or ask an LLM for advice, but first try hard to solve problems on your own, and always write up the answers entirely on your own. Make sure you understand the concepts and solutions—don't just look up the answers or use AI to answer it for you. **Cite any sources you used outside of the class material (webpages, LLMs, etc.).**

1. (Extended version of Exercise 5.3 from B&V) Consider the optimization problem

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && f(x) \leq 0, \end{aligned}$$

with $c \neq 0$. We do not assume f is convex.

- (a) Write down the dual problem in terms of the conjugate f^* . Explain why the dual problem is convex. (Don't just say it's because dual problems are always convex! Verify that the conditions for convexity are satisfied for this particular dual problem.)
- (b) Write down the dual problem for the specific constraint:

$$f(x) = x^T Q x - 1,$$

where $Q \in S_{++}^n$.

- (c) Give analytic solutions (in terms of Q and c) for x^* and λ^* for the problem in (b).
 - (d) Is your problem in part (b) strongly dual? Prove that it is or isn't.
 - (e) Again for the specific model in part (b), give an argument for why the optimal dual variable λ^* will always be strictly positive.
 - (f) Generate a random 5×5 Q matrix in Python as $Q = AA^T$, where each $A_{ij} \sim N(0, 1)$. (Note: Q will be positive-definite with probability 1.) Also, generate random $c_i \sim N(0, 1)$. Solve the primal optimization problem defined by the quadratic constraint in part (b) using CVXPY. Print out the optimal point x^* and optimal dual variable λ^* and verify they match your analytic formulas from part (c) and that λ^* is strictly positive.
 - (g) What is the value of d^* (the optimal dual value) for your solution in part (f)? How does it compare to the optimal primal value p^* ? Does this confirm your answer to part (d) about strong duality?
2. Exercise 5.14 in B&V.
 3. Exercise 5.37 in B&V. You may want to refer to the following Wikipedia page for background for this and the next problem!

https://en.wikipedia.org/wiki/Discrete-time_Markov_chain

4. A Markov chain is said to be reversible if there exists a probability vector y such that

$$p_{ij}y_j = p_{ji}y_i.$$

- (a) Show that if P is reversible with respect to a probability vector y , then y is a stationary distribution for P . That is,

$$Py = y.$$

- (b) Given any Markov chain transition matrix P and any probability vector y , we can find the nearest reversible chain with respect to y (in terms of Frobenius norm). Set this up as a convex optimization problem. (**Hint:** As we've talked about in class, this will be easier if you formulate the objective function as a *squared* Frobenius norm.)
- (c) Is this problem strongly dual? Prove that it is or isn't.
- (d) Formulate the dual problem for the optimization problem you came up with in (b). Prove that the dual objective function $g(\lambda, \nu)$ is concave. (Don't just say its because $g(\lambda, \nu)$ is always concave! Show that the specific $g(\lambda, \nu)$ function you derived here is concave.)
- (e) Solve your optimization problem in CVXPY for the specific case:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and } y = (1/3, 1/3, 1/3).$$

Find additional solutions for probability vectors:

$$y = (0.25, 0.25, 0.5), \quad \text{and } y = (1, 0, 0).$$

For all three of the solutions, verify that the resulting optimal point satisfies the feasibility constraints. Also, verify that the corresponding y is a stationary distribution (i.e., $P^*y = y$, where P^* is the optimal transition matrix that your optimization found.)

- (f) Print out the dual optimal points λ^* and ν^* that are generated by CVXPY when you solve the primal problems with the same P and three different y values in (e).
- (g) Now code up the dual problem as an optimization problem in CVXPY and solve it for the same three cases. Verify that the solutions for λ^* and ν^* are the same as reported by CVXPY as dual optimal points in (f). After observing the results of this optimization, does this confirm your answer in part (c) about whether the problem is strongly dual? Why or why not?
- (h) For each of the three y values, rerun the optimization but perturb the $x \succeq 0$ constraint to instead be $x \succeq 0.01$. In each case, how does the optimal value change? Using the optimal dual variables, explain why each perturbed solution behaves the way it does. (Hint: one solution is particularly not like the others! Also, explain why it behaves so differently.)