ECE 4750/6750 + CS 4501/6501: Digital Signal Processing

Final Exam Practice Problems Solutions

1. For the following signals, write if they are "periodic" or "aperiodic". If they are periodic, also give their frequency and period.

   (a) $x[n] = \cos\left(\frac{4n}{5}\right)$

   Aperiodic

   (b) $x[n] = \sin\left(\frac{12\pi n}{7}\right)$

   Periodic, with frequency $\frac{12\pi}{7}$ and period 7.

   (c) $x[n] = e^{i3\pi n}$

   Periodic, with frequency 3π and period 2.

   (d) $x[n] = 3^n e^{\frac{2\pi n}{3}}$

   Aperiodic

2. Consider the following systems:

$$T_1\{x[n]\} = x[2n+1] \qquad T_2\{x[n]\} = \sin(x[n]) \qquad T_3\{x[n]\} = 3$$

$$T_4\{x[n]\} = \sum_{k=-\infty}^{n} x[k] \qquad T_5\{x[n]\} = \sum_{k=1}^{\infty} \frac{1}{2^k} x[n-k] \qquad T_6\{x[n]\} = x[n] * x[n]$$

   (a) List which of these systems are linear.

   T_1, T_4, T_5

   (b) List which of these systems are time-invariant.

   T_2, T_3, T_4, T_5, T_6

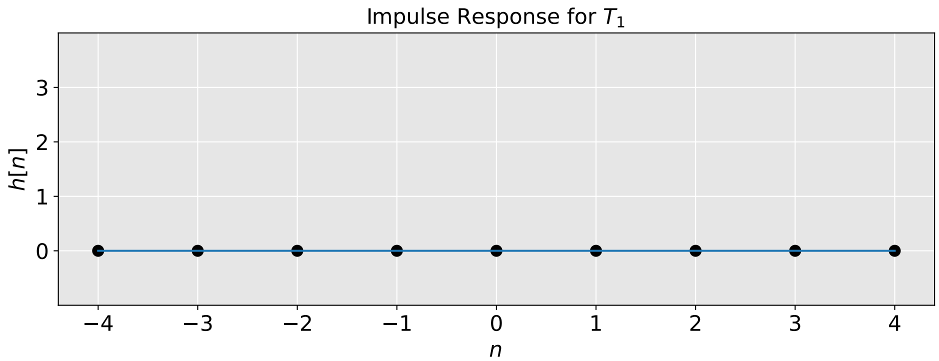
   (c) List which of these systems are LTI and BIBO stable.

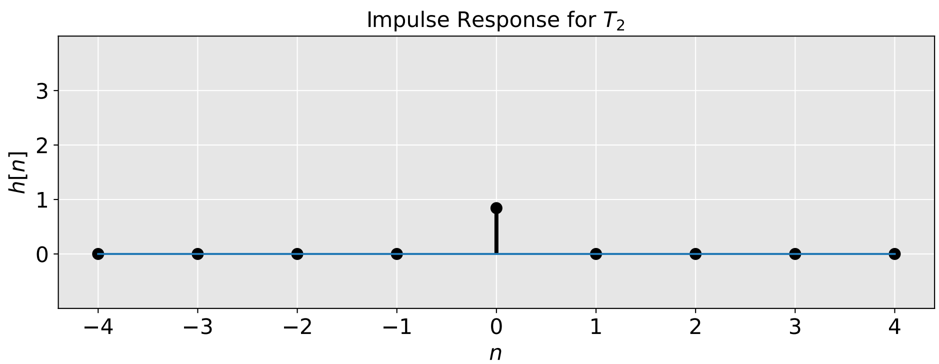
   T_5

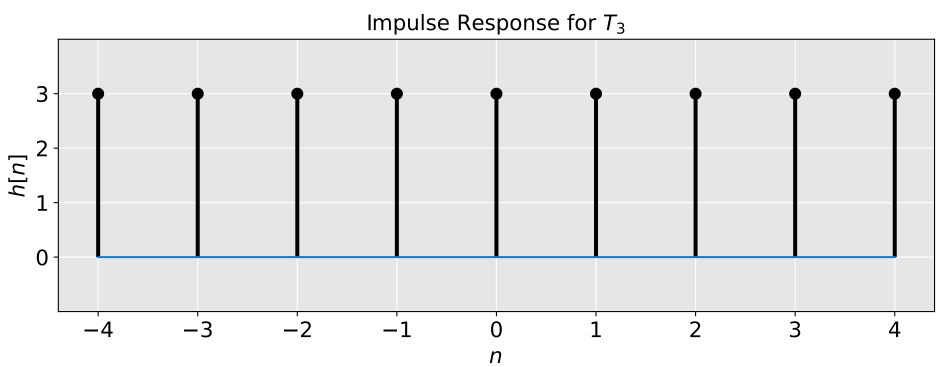
   (d) List which of these systems are causal.

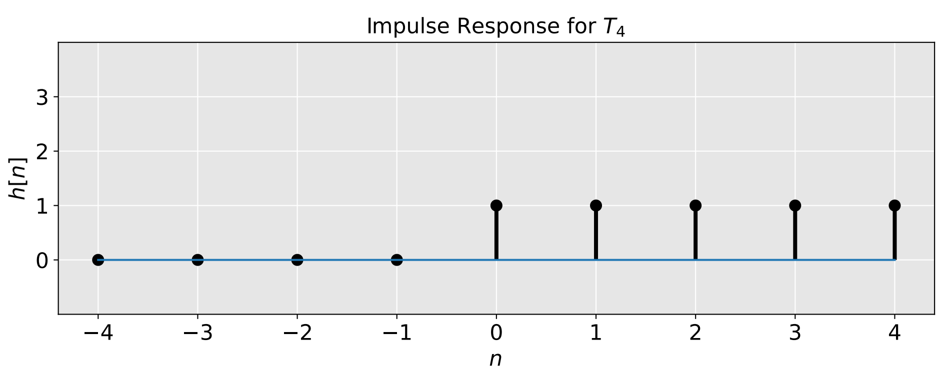
   T_2, T_3, T_4, T_5

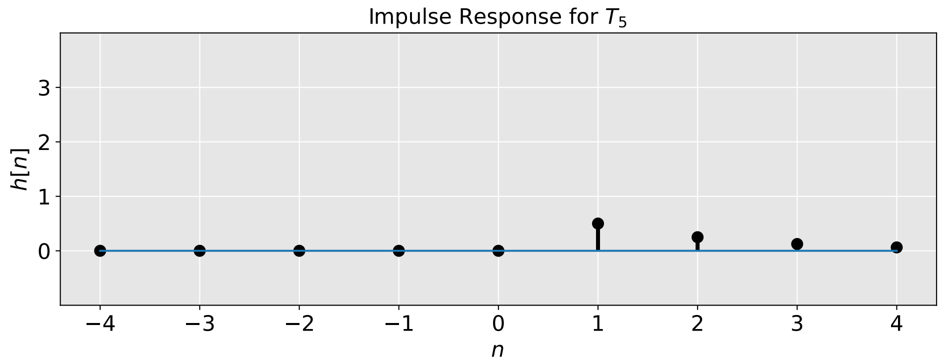
   (e) For each system, sketch the impulse response function for each of these systems in the range $-4 \leq n \leq 4$.

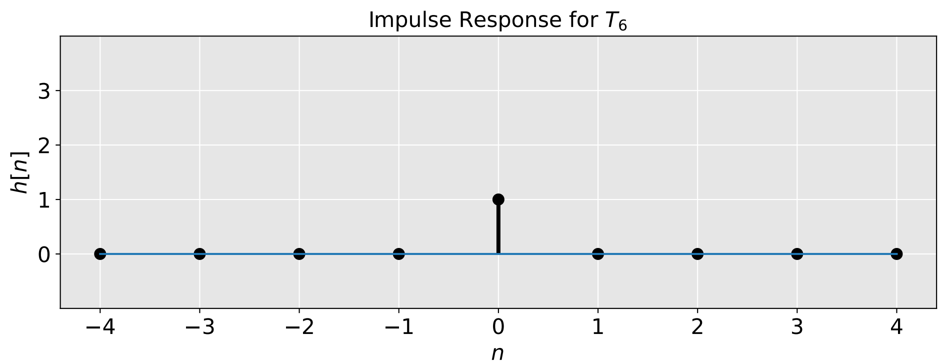

Impulse Response for T_1
$h[n]$
3
2
1
0
−4
−3
−2
−1
0
1
2
3
4
n


Impulse Response for T_2
$h[n]$
3
2
1
0
−4
−3
−2
−1
0
1
2
3
4
n


Impulse Response for T_3
$h[n]$
3
2
1
0
−4
−3
−2
−1
0
1
2
3
4
n


Impulse Response for T_4
$h[n]$
3
2
1
0
−4
−3
−2
−1
0
1
2
3
4
n





3. Consider two periodic signals, $x_1[n]$ and $x_2[n]$, both with period $L = 10$, given by:

$$x_1[n] = \cos(\pi n), \qquad x_2[n] = \sin(\pi n).$$

(a) What is ω_0 for these two signals?

$\frac{2\pi}{L} = \frac{\pi}{5}$

(b) What are their DFTs, $X_1[k]$ and $X_2[k]$?

$X_1[k] = \sqrt{10}\delta[k-5], \qquad X_2[k] = 0$

(c) What is $\mathcal{DFT}\{x_1[n] * x_2[n]\}$? This should be a function of k only. Write out all known constants (for example, don't use the symbol L, but rather plug in its value $L = 10$, etc.).

$\mathcal{DFT}\{x_1[n] * x_2[n]\} = X_1[k]X_2[k] = 0$

(d) What is $x_1[n] * x_2[n]$? Again,this should be a function of k, and write out all known constants rather than leaving the corresponding symbols.

$x_1[n] * x_2[n] = 0$

(e) If these signals came from sampling a continuous signal with sampling period $T = 0.04$ seconds, what is the frequency in Hertz represented by the bin $k = 3$ in the DFT?

The bin frequency is $\frac{k}{LT} = \frac{3}{10\times 0.04} = 7.5$ Hz.

4. Consider the following LCCDE:

$$y[n] = x[n] + 2.0x[n-1] + y[n-1] - \frac{1}{2}y[n-2]$$

(a) Write down the transfer function, $H(z)$, for this system.

First, take z-transform of both sides to get:

$$Y(z) = X(z) + 2.0z^{-1}X(z) + z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z).$$

Now, collect $Y(z)$ and $X(z)$ terms to get:

$$Y(z)(1 - z^{-1} + \frac{1}{2}z^{-2}) = X(z)(1 + 2z^{-1}).$$

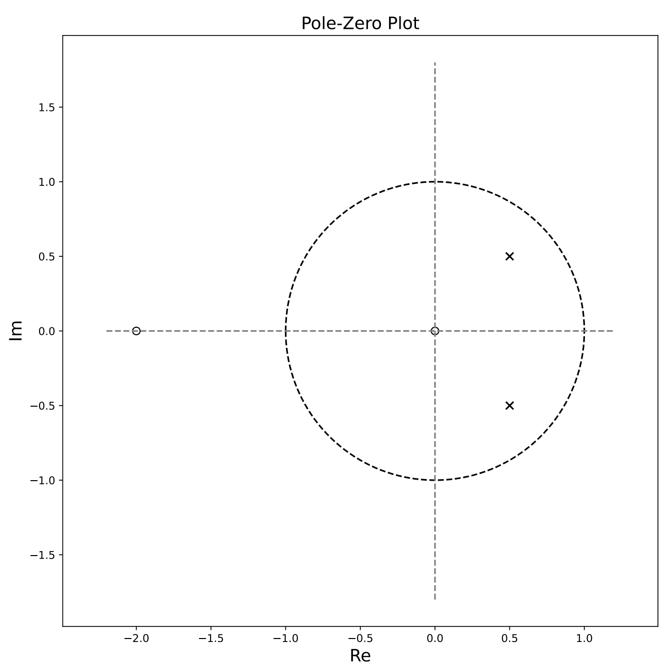
Now write the transfer function as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}.$$

(b) Sketch a pole-zero plot for this system.

First, get the poles and zeros by factoring the polynomials in numerator and denominator:

$$H(z) = \frac{z^2}{z^2}\frac{1 + 2z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \frac{z^2 + 2z}{z^2 - z + \frac{1}{2}} = \frac{z(z+2)}{(z - 0.5 - 0.5i)(z - 0.5 + 0.5i)}.$$



(c) Is this a causal system?

Yes, the LCCDE only uses timepoints $\leq n$.

(d) What is the region of convergence for this system?

Outside the radius of the poles, i.e., $|z| \geq |0.5 + 0.5i| = \frac{\sqrt{2}}{2}$.

(e) Is this a BIBO stable system?

Yes, ROC contains the unit circle.

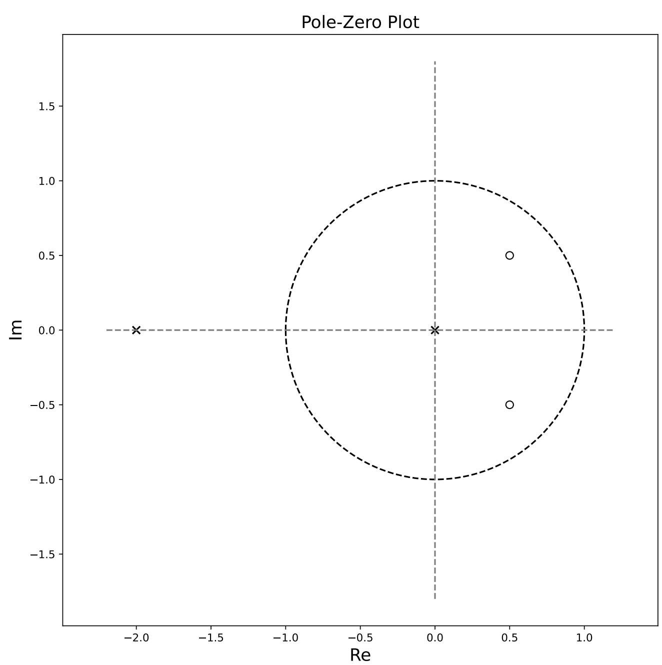
(f) Which best describes the frequency response: low-pass, high-pass, band-pass, or all-pass? Explain why.

Although this system does not suppress any frequencies, it will amplify the frequencies near the poles, which are at $\pm\frac{\pi}{4}$. So, it could be considered a band-pass filter. (Would also accept "none of the above" here because again, it doesn't suppress any frequencies.)

(g) What is transfer function for the inverse system?

$$H^{-1}(z) = \frac{1 - z^{-1} + \frac{1}{2}z^{-2}}{1 + 2z^{-1}}.$$

(h) Sketch a pole-zero plot for the inverse transfer function.



(i) What region of convergence makes the inverse BIBO stable? (say "none" if it isn't possible)

$$0 < |z| < 2$$

(j) Is this BIBO stable system also causal? (say "not applicable" if previous answer was "none")

No, this is not causal because the ROC does not extend to infinity.

(k) Write an LCCDE corresponding to the inverse transfer function. (There are multiple correct answers depending on the ROC you choose.)

Reading off of the system function for $H^{-1}(z)$, we have:

$$y[n] + 2y[n-1] = x[n] - x[n-1] + \frac{1}{2}x[n-2].$$

5. Consider two continuous signals, $x_1(t), x_2(t)$, with corresponding Fourier transforms $X_1(\Omega), X_2(\Omega)$, that satisfy:

$$|X_1(\Omega)| = 0, \text{for } |\Omega| > 12\pi, \qquad |X_2(\Omega)| = 0, \text{for } |\Omega| > 5\pi.$$

(a) Assume both signals were sampled with a time period of $T = 0.1$ seconds. Is $x_1(t)$ being sampled above the Nyquist rate? Is $x_2(t)$?

The angular sampling rate is $\frac{2\pi}{T} = 20\pi$. This is lower than $2\times$ the maximum angular frequencies in x_1 and higher than $2\times$ the maximum frequencies in x_2. So x_1 would not be sampled above the Nyquist rate, whereas x_2 would be.

(b) Consider the signal $x_1(t) * x_2(t)$. What is the Nyquist sampling rate (in Hertz) needed to faithfully represent it?

We know $\mathcal{F}\{x_1(t) * x_2(t)\} = \mathcal{F}\{x_1(t)\} \times \mathcal{F}\{x_2(t)\} = X_1(\Omega)X_2(\Omega)$. So, this will be zero if either $X_1(\Omega)$ or $X_2(\Omega)$ is. In other words, if $|\Omega| > 5\pi$. So, the Nyquist rate is $f_s > 2 \times \frac{5\pi}{2\pi} = 5$ Hz.

(c) Consider the signal $x_1(t)x_2(t)$. What is the Nyquist sampling rate (in Hertz) needed to faithfully represent it?

We know $\mathcal{F}\{x_1(t)x_2(t)\} = \mathcal{F}\{x_1(t)\} * \mathcal{F}\{x_2(t)\} = X_1(\Omega) * X_2(\Omega)$. The extents of the domains where X_1 and X_2 are non-zero will add together. Therefore, $X_1(\Omega) * X_2(\Omega)$ will be zero if $|\Omega| > 12\pi + 5\pi = 17\pi$. Then the Nyquist frequency is $f_s > 2 \times \frac{17\pi}{2\pi} = 17$ Hz.

6. Consider a stable LTI system with transfer function

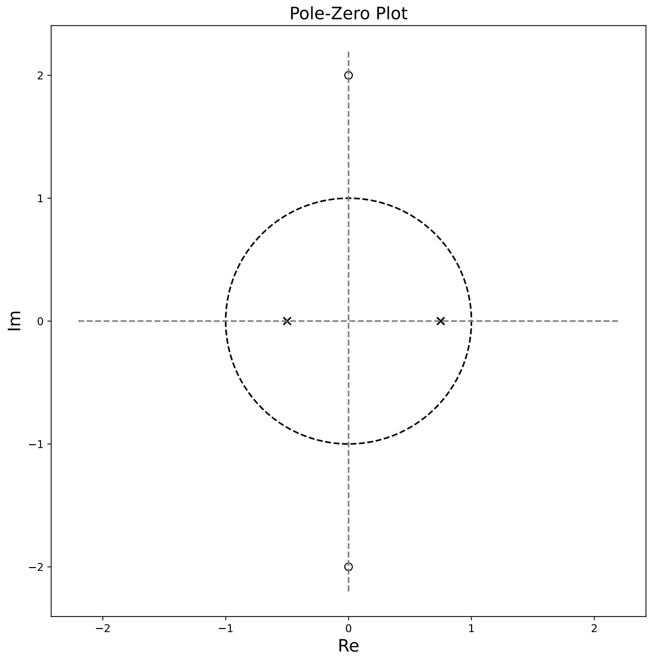
$$H(z) = \frac{1 + 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

(a) Sketch a pole-zero diagram for this system.

First, we'll factor the transfer function:

$$H(z) = \frac{z^2 + 4}{z^2 - \frac{1}{4}z - \frac{3}{8}} = \frac{(z - 2i)(z + 2i)}{(z - \frac{3}{4})(z + \frac{1}{2})}.$$

This gives us the following poles and zeros:



(b) What is the ROC for this system? Sketch it on the pole-zero diagram.

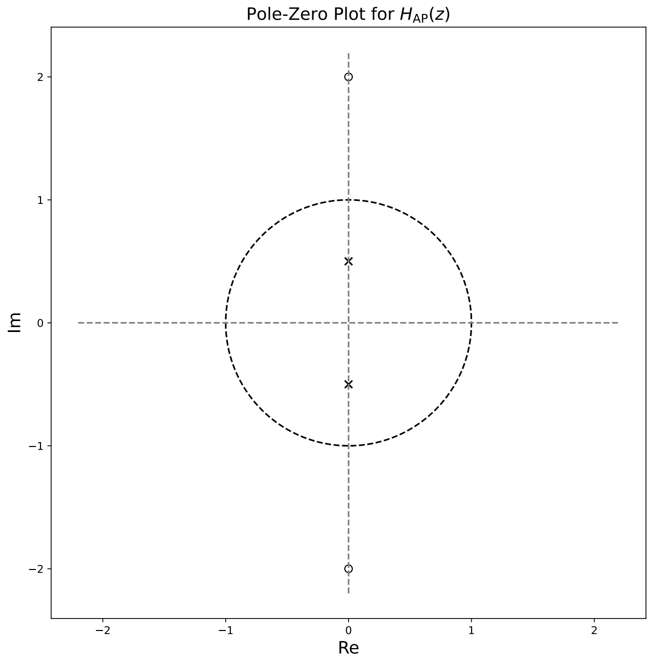
Because it is a stable system, we know that it must contain the unit circle. So, the ROC will be the everything outside the circle of the outermost pole (0.75), that is, $|z| > 0.75$.

(c) Factorize this system into a product of an all-pass system, $H_{\text{AP}}(z)$, and a minimum-phase system, $H_{\min}(z)$, i.e., $H(z) = H_{\text{AP}}(z)H_{\min}(z)$. Write equations for the system functions $H_{\text{AP}}(z)$ and $H_{\min}(z)$.

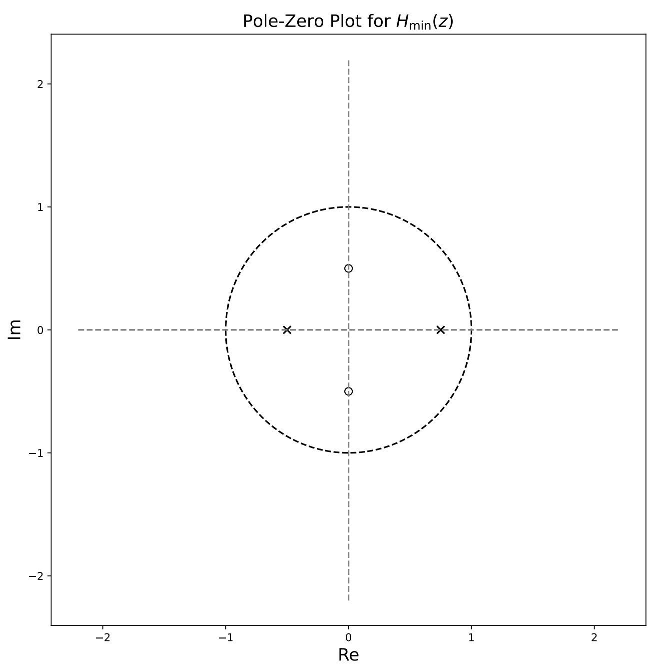
We know that the all-pass system must have each zero c matched with a conjugate and inverse pole $\bar{c}^{-1}$. We know the minimum phase system has all poles and zeros inside the unit circle. For this system, the zeros are outside the unit circle, so they have to be in the all-pass factor. The poles are inside the unit circle, and they can go in the minimum-phase factor. We'll have to add conjugate-inverse pole pairs to the all-pass system, and these will have to be canceled by matching zeros in the minimum-phase system. All together we have:

$$H_{\text{AP}}(z) = \frac{(z+2i)(z-2i)}{(z-0.5i)(z+0.5i)},$$
$$H_{\min}(z) = \frac{(z-0.5i)(z+0.5i)}{(z-\frac{3}{4})(z+\frac{1}{2})}.$$

(d) Draw pole-zero diagrams for $H_{\text{AP}}(z)$ and $H_{\min}(z)$. Sketch the ROC for both systems in the diagrams.



The ROC will be everything outside the circle of the poles, $|z| > 0.5$.



The ROC will be everything outside the circle of the outermost pole, $|z| > 0.75$.

7. Determine if the following statements are true. If a statement is true, give a concise arguments for why. If it is false, give a system that provides a counterexample.

   (a) The transfer function for a causul, finite impulse response filter must have a pole at zero with multiplicity k, where k is the largest integer for which there is an $x[n-k]$ term in the corresponding LCCDE.

This is **true**. Any causal FIR filter will be of the form:

$$H(z) = a_0 + a_1 z^{-1} + \cdots + a_k z^{-k}.$$

Multiplying this by $\frac{z^k}{z^k} = 1$ gives us:

$$H(z) = \frac{a_0 z^k + a_1 z^{k-1} + \cdots + a_k}{z^k}.$$

The denominator indicates that there is a pole at $z = 0$ with multiplicity k.

(b) The transfer function for an infinite impulse response filter must have a pole somewhere in the z-plane away from the origin.

This is also **true**. For an LCCDE to be IIR it has to contain feedback, so the denominator of $H(z)$ will contain a factor of the form $(z - a)$, where $a \neq 0$ is a pole.

(c) A finite impulse response filter will always have linear phase.

This is **false**. Any impulse response that doesn't have symmetry in time will result in non-linear phase. For example, $h[n] = \delta[n] + 2\delta[n-1]$.

(d) The transfer function for a zero-phase filter has poles only at the origin (or no poles).

This is **false**. We can have an IIR filter with an even impulse response, that is, $h[n] = h[-n]$. For example, take the impulse response:

$$h[n] = 0.5^n u[n] + 0.5^{-n} u[-n-1].$$

This exponentially decays to the left and right. The corresponding $H(z)$ has phase zero.

Note: The above answer would be sufficient, but here are some extended details about this system:

To get the $H(z)$, take the z-transform of the left- and right-sided exponentials separately:

$$\mathcal{Z}\{0.5^n u[n]\} = \sum_{n=-\infty}^{-1} 0.5^{-n} z^{-n} = \sum_{m=1}^{\infty} (0.5z)^m = \frac{0.5z}{1 - 0.5z}.$$

$$\mathcal{Z}\{0.5^{-n} u[-n-1]\} = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \sum_{n=0}^{\infty} (0.5z^{-1})^n = \frac{1}{1 - 0.5z^{-1}}.$$

Then add them together to get:

$$H(z) = \frac{0.5z(z - 0.5) + z(1 - 0.5z)}{(z - 0.5)(1 - 0.5z)} = \frac{0.75z}{(z - 0.5)(1 - 0.5z)}.$$

We can verify that this has poles at 0.5 and 2. To verify that this system is zero-phase, we can further simplify $H(z)$ to get:

$$H(z) = \frac{0.75}{(z - 0.5)(z^{-1} - 0.5)} = \frac{0.75}{1 - 0.5z - 0.5z^{-1} + 0.25}.$$

Using the fact that $z^{-1} = \bar{z}$ when z is on the unit circle ($z = e^{i\omega}$), we see that the denominator would be real-valued, and thus $H(e^{i\omega})$ would be real-valued, in other words, zero-phase.