Bayesian Estimation

Foundations of Data Analysis

February 24, 2022

All models are wrong, but some are useful.

— George Box

Frequentist vs. Bayesian Statistics

Frequentist:

$$L(\theta\,;\,x_1,\ldots,x_n) = \prod_{i=1}^{n} p(x_i\,;\,\theta)$$

Bayesian:

$$p(\theta \mid x_1,\ldots,x_n) = \frac{p(x_1,\ldots,x_n \mid \theta)p(\theta)}{p(x_1,\ldots,x_n)}$$

Frequentist vs. Bayesian Statistics

Frequentist: θ is a parameter

$$L(\theta\,;\,x_1,\ldots,x_n) = \prod_{i=1}^{n} p(x_i\,;\,\theta)$$

Bayesian: θ is a random variable

$$p(\theta \mid x_1,\ldots,x_n) = \frac{p(x_1,\ldots,x_n \mid \theta)p(\theta)}{p(x_1,\ldots,x_n)}$$

Why is Random θ Important?

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- We can make **probabilistic statements** about θ (e.g., mean, variance, quantiles, etc.).
- If θ is one of several competing **hypotheses**, we can assign it a probability.
- We can make **probabilistic predictions** of the next data point, $\hat{x}$, using

$$p(\hat{x} \mid x_1, \ldots, x_n) = \int p(\hat{x} \mid \theta)\, p(\theta \mid x_1, \ldots, x_n)\, d\theta$$

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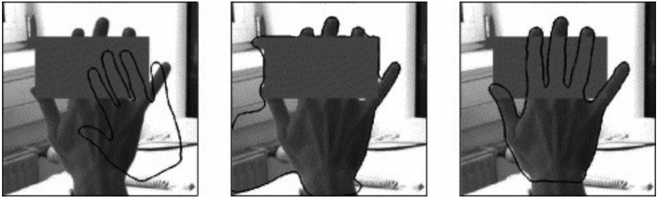
- Whether using frequentist or Bayesian models, **always check the assumptions you make.**
- Sometimes prior knowledge is a good thing.



Cremers, et al., Pattern Recognition, 2003

Deductive Logic

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Logic	Probability
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$$P(A \mid B) > P(A) \qquad \text{Bayes' Rule}$$

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$$\frac{\begin{array}{l}\text{If } A \text{ is true, } B \text{ becomes more likely.} \\ B \text{ is true.}\end{array}}{A \text{ is more likely.}}$$

If it's raining, then the sidewalk is more likely to be wet.

Flipping the implication: $P(B \mid A) > P(B)$

If A is true, B becomes more likely.

B is true.

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The sidewalk is wet.

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If it's raining, then the sidewalk is more likely to be wet.

The sidewalk is wet.

It's more likely to be raining.

Exercise for You

Given that $P(B \mid A) > P(B)$, show that:

1. If $\bar{B}$ happens, A becomes less likely.
   (weak form of *modus tollens*)
2. If $\bar{A}$ happens, B becomes less likely.

Final Bayesian Logic Rules

Given that $P(B \mid A) > P(B)$, analagous to $A \Rightarrow B$, we have four rules:

1. If A, then B is more likely (weak *modus ponens*)
2. If $\bar{B}$, then A is less likely (weak *modus tollens*)
3. If B, then A is more likely (no logical equivalent)
4. If $\bar{A}$, then B is less likely (no logical equivalent)

Cold Example

		C = 0	C = 1	
R	0	0.50	0.05	0.55
	1	0.20	0.25	0.45
		0.7	0.3	

Remember:

$P(C) = 0.3$

$P(C \mid R) = 0.56$

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Remember:
$P(C) = 0.3$
$P(C \mid R) = 0.56$

What if I didn't give you the full table, but just:

$$P(R \mid C) = 0.83 \quad > \quad P(R) = 0.45$$

What can you say about the increase $P(C \mid R) > P(C)$?

Cold Example

Notice, having a cold *increases* my chance for a runny nose by the factor,

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MLE of Bernoulli Proportion

$$X \sim \mathrm{Ber}(\theta)$$

$$L(\theta \mid x_1, \ldots, x_n) = \theta^k (1-\theta)^{n-k}, \quad \text{where } k = \sum_i x_i$$

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\end{aligned}$$

$$\frac{dL}{d\theta}\left(\hat{\theta}\right) = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{k}{n}$$

Bayesian Inference of a Bernoulli Proportion

Let's give θ a uniform prior: $\theta \sim \mathrm{Unif}(0, 1)$

Posterior:

$$p(\theta \,|\, x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n \,|\, \theta) p(\theta)}{p(x_1, \ldots, x_n)}$$

$$= \frac{p(x_1, \ldots, x_n \,|\, \theta)}{p(x_1, \ldots, x_n)}$$

Bayesian Inference of a Bernoulli Proportion

Just need the denominator (normalizing constant):

$$\begin{aligned} p(x_1, \ldots, x_n) &= \int_0^1 p(x_1, \ldots, x_n \mid \theta) p(\theta) d\theta \\ &= \int_0^1 \theta^k (1-\theta)^{n-k} d\theta \\ &= \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \end{aligned}$$

Resulting posterior is:

$$p(\theta \mid x_1, \ldots, x_n) = \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} \theta^k (1-\theta)^{n-k}$$

Beta Distribution

$X \sim \mathrm{Beta}(\alpha, \beta)$ PDF:

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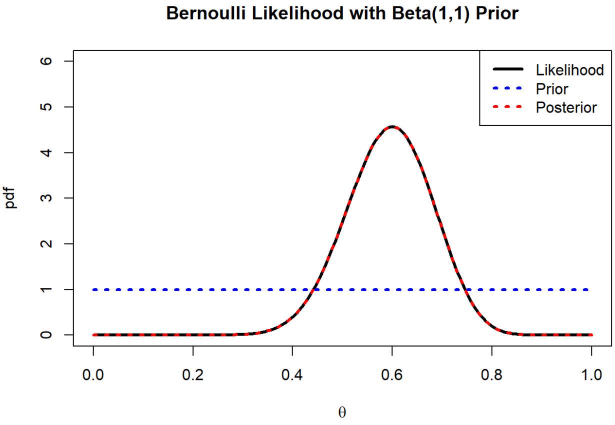
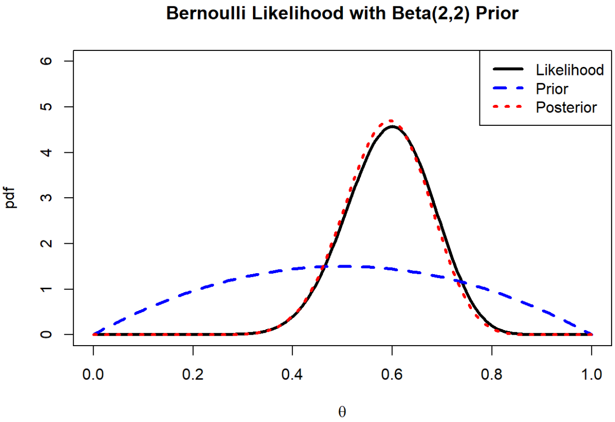
So, posterior of Bernoulli with Uniform prior is $\theta \sim \mathrm{Beta}(k + 1, n - k + 1)$.

Example

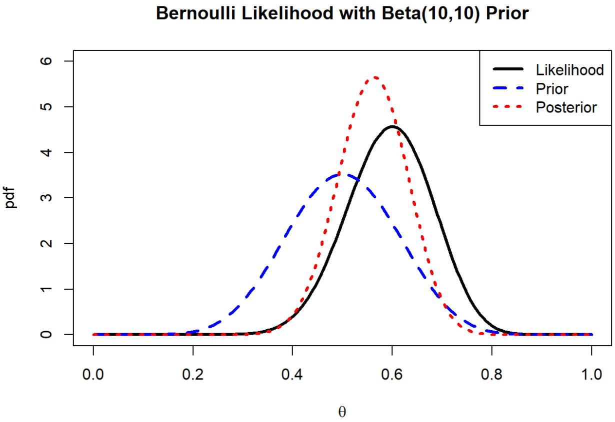

Bernoulli Likelihood with Beta(1,1) Prior
pdf
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
θ
Likelihood
Prior
Posterior

Example

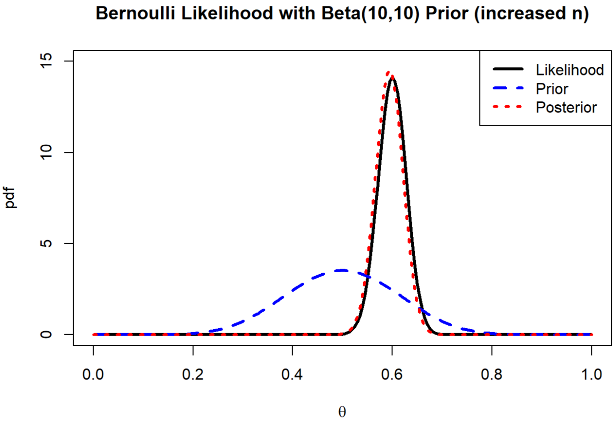


Bernoulli Likelihood with Beta(2,2) Prior
Likelihood
Prior
Posterior
pdf
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
θ

Example


Bernoulli Likelihood with Beta(10,10) Prior
pdf
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
θ
Likelihood
Prior
Posterior

Example


Bernoulli Likelihood with Beta(10,10) Prior (increased n)
Likelihood
Prior
Posterior
pdf
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0
θ

Laplace’s Analysis of Birth Rates

Mémoire sur les probabilités (1778)

`http://cerebro.xu.edu/math/Sources/Laplace/`

Problem: Boys were born at a consistently, but only slightly, higher rate than girls in Paris. Was this a real effect or just due to chance?

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Boys: $k = 251527$ # Girls: $n - k = 241945$

Solution: Model the proportion of boys as the posterior: $\theta \,|\, k \sim \mathrm{Beta}(251528, 241946)$. Then,

$$P(\theta \leq 0.5 \,|\, k) = F_{\theta|k}(0.5) = 1.15 \times 10^{-42}$$